$f(t) = f_c + k_f m(t)$ — max: $f_c + k_f m_p$; min: $f_c - k_f m_p$

5.3 Bandwidth of FM Signals

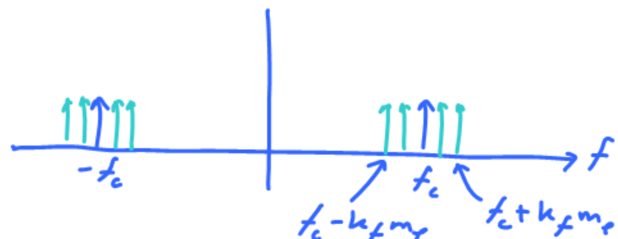
5.24. FM: The "Holy Grail" Technique for BW Saving?

In the 1920s, the idea of frequency modulation (FM) was naively proposed very early as a method to conserve the radio spectrum. The argument was presented as follows:

- If $m(t)$ is bounded between $-m_p$ and m_p, then the maximum and minimum values of the (instantaneous) carrier frequency would be $f_c+k_f m_p$ and $f_c - k_f m_p$, respectively. (Think of this as a delta function shifting to various location between $f_c + k_f m_p$ and $f_c - k_f m_p$ in the frequency domain.)

Wrong argument!

$BW = 2k_f m_p$



- Hence, the spectral components would remain within this band with a bandwidth $2km_p$ centered at f_c.
- Conclusion: By using an arbitrarily small k, we could make the information bandwidth arbitrarily small (much smaller than the bandwidth of $m(t)$.

In 1922, Carson argued that this is an ill-considered plan. We will illustrate his reasoning later. In fact, experimental results shows that

BW of FM $\geq$ BW of AM

As a result of his observation, FM temporarily fell out of favor.

5.25. Armstrong (1936) reawakened interest in FM when he realized it had a much different property that was desirable. When the k_f is large, the inverse mapping from the modulated waveform $x_{\mathrm{FM}}(t)$ back to the signal $m(t)$ is much less sensitive to additive noise in the received signal than is the case for amplitude modulation. FM then came to be preferred to AM because of its higher fidelity. [1, p 5-6]

Finding the "bandwidth" of FM Signals turns out to be a difficult task. Here we present a few approximation techniques.

5.26. First, from 5.20, we see that both FM and PM can be viewed as

$$x(t) = A\cos\left(2\pi f_c t + \theta_0 + \phi(t)\right) \tag{71}$$

$e^{jx} = \cos x + j\sin x$

$\cos x = \mathrm{Re}\{e^{jx}\}$

where $\phi(t) = (m * h)(t)$ if $h(t)$ is selected properly.

The Fourier transform of $\phi(t)$ is $\Phi(f) = M(f)H(f)$. So, if $M(f)$ is band-limited to B, we know that $\Phi(f)$ is also band-limited to B as well.

Now, let us rewrite (71) as

$$x(t) = A\,\mathrm{Re}\left\{e^{j(2\pi f_c t + \theta_0 + \phi(t))}\right\} = A\,\mathrm{Re}\left\{e^{j(2\pi f_c t + \theta_0)}e^{j\phi(t)}\right\} \tag{72}$$

Recall that Taylor series expansion of e^z around $z = 0$ is

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Plugging in $z = j\phi(t)$ gives

$$e^{j\phi(t)} = 1 + j\phi(t) + \frac{(j\phi(t))^2}{2!} + \frac{(j\phi(t))^3}{3!} + \cdots = 1 + j\phi(t) - \frac{\phi^2(t)}{2!} + (-j)\frac{\phi^3(t)}{3!} + \cdots \tag{73}$$

Applying the Euler's formula

$$e^{j(2\pi f_c t + \theta_0)} = \cos\left(2\pi f_c t + \theta_0\right) + j\sin\left(2\pi f_c t + \theta_0\right)$$

and (73) to (72) gives

$$x(t) = A\left(\cos\left(2\pi f_c t + \theta_0\right) - \phi(t)\sin\left(2\pi f_c t + \theta_0\right) - \frac{\phi^2(t)}{2!}\cos\left(2\pi f_c t + \theta_0\right) + \frac{\phi^3(t)}{3!}\sin\left(2\pi f_c t + \theta_0\right) + \cdots\right).$$

Recall that if $\phi(t)$ is band-limited to B, then $\phi^n(t)$ is band-limited to nB. With such series, there is no bound for the value of n and therefore, we conclude that the absolute bandwidth would be infinite.

$e^x \approx 1 + x$ when x is small

5.27. Narrowband Angle Modulation: When $\phi(t)$ is small, we may approximate e^z by $z + 1$. Therefore,

$$e^{j\phi(t)} \approx 1 + j\phi(t). \tag{74}$$

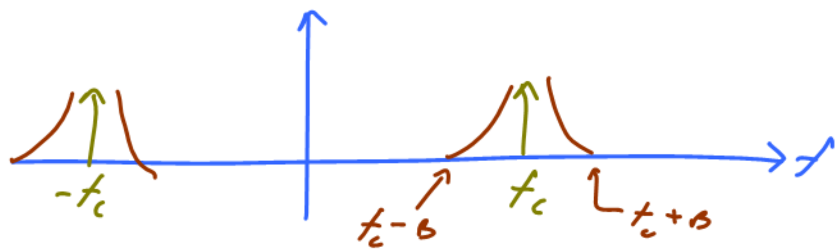
Applying the Euler's formula

$$e^{j(2\pi f_c t + \theta_0)} = \cos\left(2\pi f_c t + \theta_0\right) + j\sin\left(2\pi f_c t + \theta_0\right)$$

and (74) to (72) gives

$$x(t) = A\,\mathrm{Re}\left\{e^{j(2\pi f_c t+\theta_0)}e^{j\phi(t)}\right\}$$

$$\begin{aligned} &\approx A\,\mathrm{Re}\left\{\left(\cos\left(2\pi f_c t+\theta_0\right)+j\sin\left(2\pi f_c t+\theta_0\right)\right)\left(1+j\phi(t)\right)\right\} \\ &= \underline{A\cos\left(2\pi\right.}\left.f_c t+\theta_0\right) - A\underline{\phi(t)\sin\left(\right.}\left.2\pi f_c t+\theta_0\right) \end{aligned}$$



- The "approximated" expression of $x(t)$ is similar to AM.
  - The first term yields a carrier component. The second term generates a pair of sidebands. Thus, if $\phi(t)$ has a bandwidth B, the bandwidth of $x(t)$ is $2B$.
- The important difference between AM and angle modulation is that the sidebands are produced by multiplication of the message-bearing signal, $\phi(t)$, with a carrier that is in phase quadrature with the carrier component, whereas for AM they are not.
- The FM signal whose $\left|2\pi k_f \int\limits_{-\infty}^{t} m(\tau)d\tau\right| \ll 1$ is called **narrowband FM (NBFM)**. The PM signal whose $|k_p m(t)| \ll 1$ is called **narrowband PM (NBPM)**. Note that these conditions are satisfied when $k_f \ll 1$ or $k_p \ll 1$, respectively. [5, p 260]
- For larger values of $|\phi(t)|$ the terms $\phi^2(t)$, $\phi^3(t)$, …in (73) cannot be ignored and will increase the bandwidth of $x(t)$.
- Recall, from (32) that

$$g(t)\cos(2\pi f_c t+\phi) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(e^{j\phi}G(f-f_c)+e^{-j\phi}G(f+f_c)\right).$$

Therefore, when

$$x(t) \approx A\cos\left(2\pi f_c t+\theta_0\right) - A\phi(t)\cos\left(2\pi f_c t+\theta_0-90^\circ\right),$$

we have

$$X(f) \approx \frac{A}{2}\left(e^{j\theta_0}\delta(f-f_c) + e^{-j\theta_0}\delta(f+f_c) - e^{j(\theta_0-90^\circ)}\Phi(f-f_c) - e^{-j(\theta_0-90^\circ)}\Phi(f+f_c)\right)$$
$$= \frac{A}{2}\left(e^{j\theta_0}\delta(f-f_c) + e^{-j\theta_0}\delta(f+f_c) + je^{j\theta_0}\Phi(f-f_c) - je^{-j\theta_0}\Phi(f+f_c)\right).$$

5.28. Wideband FM (WBFM): For potentially wideband $m(t)$, here, we present a technique to roughly estimate the bandwidth of $x_{FM}(t)$.

To do this, we consider $m(t)$ that is a piecewise constant function (also known as step function or staircase function); this implies that the instantaneous frequency $f(t) = f_c + k_f m(t)$ of $x_{\mathrm{FM}}(t)$ is also piecewise constant as shown in Figure 36.

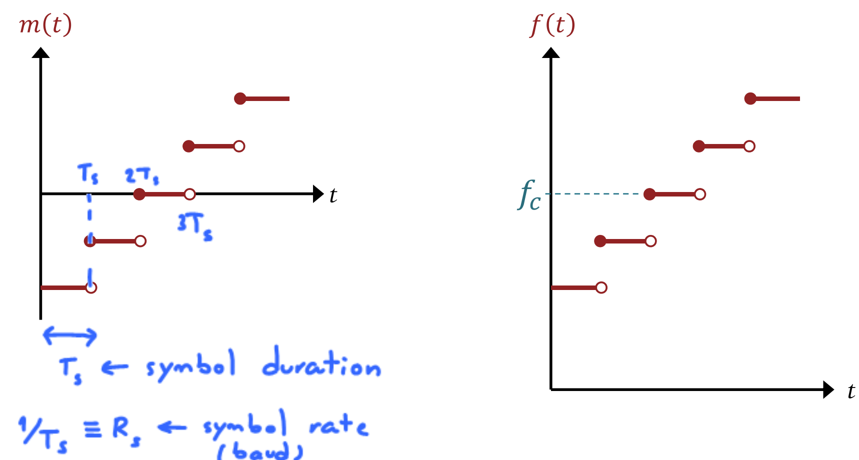


Figure 36: FM for discrete-valued (digital) message

For example, we can consider the transmitted signal $x_{\mathrm{FM}}(t)$ constructed from five different tones. Its instantaneous frequency is increased from f_1 to f_5.

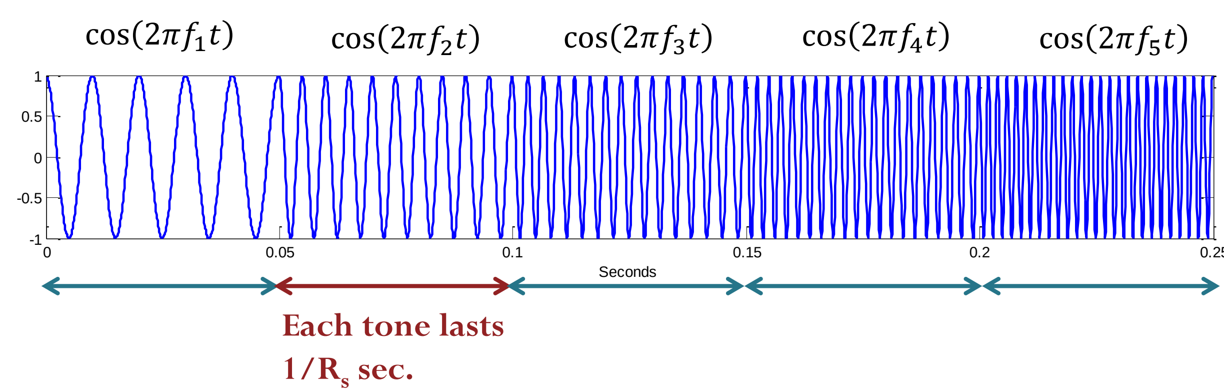


Figure 37: $x_{\mathrm{FM}}(t)$ for discrete-valued (digital) message in Figure 36.

Assume that each tone lasts $T_s = \frac{1}{R_s}$ [s] where R_s is called the "(symbol) rate" of the data transmission. The value of R_s indicates how fast the values of $m(t)$ is changed. Increasing the value of R_s reduces the time to complete the transmission.

Recall that the Fourier transform of a cosine contains simply (two shifted and scaled) delta functions at the (plus and minus) frequency of the cosine. However, recall also that when we consider the cosine pulse, which is time-limited, its Fourier transform contains (two) sinc functions. In particular, the cosine pulse

$$p(t) = \begin{cases} \cos(2\pi f_0 t), & t_1 \le t < t_2, \\ 0, & \text{otherwise}, \end{cases}$$

can be viewed as the pure cosine function $\cos(2\pi f_0 t)$ multiplied by a rectangular pulse $r(t) = 1[t_1 \le t < t_2]$. By (31), we know that multiplication by $\cos(2\pi f_0 t)$ will shift the spectrum $R(f)$ of the rectangular pulse to $\pm f_c$ and scaled its values by a factor of $\frac{1}{2}$: $P(f) = \frac{1}{2}R(f - f_0) + \frac{1}{2}R(f + f_0)$

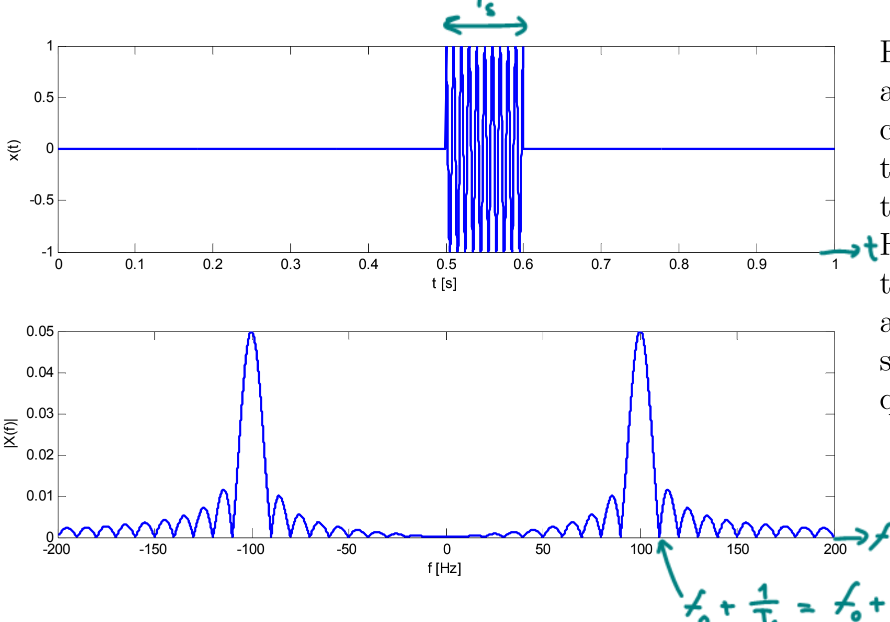


Figure 38: Cosine pulse and its spectrum which contains two sinc functions at $\pm$ freqeuncy of the cosine (which is 100 Hz in the figure). When the pulse only lasts for a short time period, the sinc pulses in the frequency domain are wide.

where the Fourier transform[24] $R(f)$ of the rectangular pulse is given by

$$R(f) = (t_2 - t_1)\, e^{-j\pi f(t_1+t_2)}\, \mathrm{sinc}\,(\pi f (t_2 - t_1)).$$

[24]To get this, first consider the rectangular pulse of width $t_2 - t_1$ centered at t = 0. From (15), the corresponding Fourier transform is $2\left(\frac{t_2-t_1}{2}\right)\mathrm{sinc}\left(2\pi\left(\frac{t_2-t_1}{2}\right)f\right)$. Finally, by time-shifting the rectangular pulse in the time domain by $\frac{t_2+t_1}{2}$, we simply multiply the Fourier transform by $e^{-2\pi f\left(\frac{t_2-t_1}{2}\right)}$ in the frequency domain.

See Figure 38 for an example.

When $m(t)$ is piecewise constant, $x_{\text{FM}}(t)$ is a sum of cosine pulses. Therefore, its spectrum $X(f)$ will be the sum of the sinc functions centered at the frequencies of the pulses as shown in Figure 39.

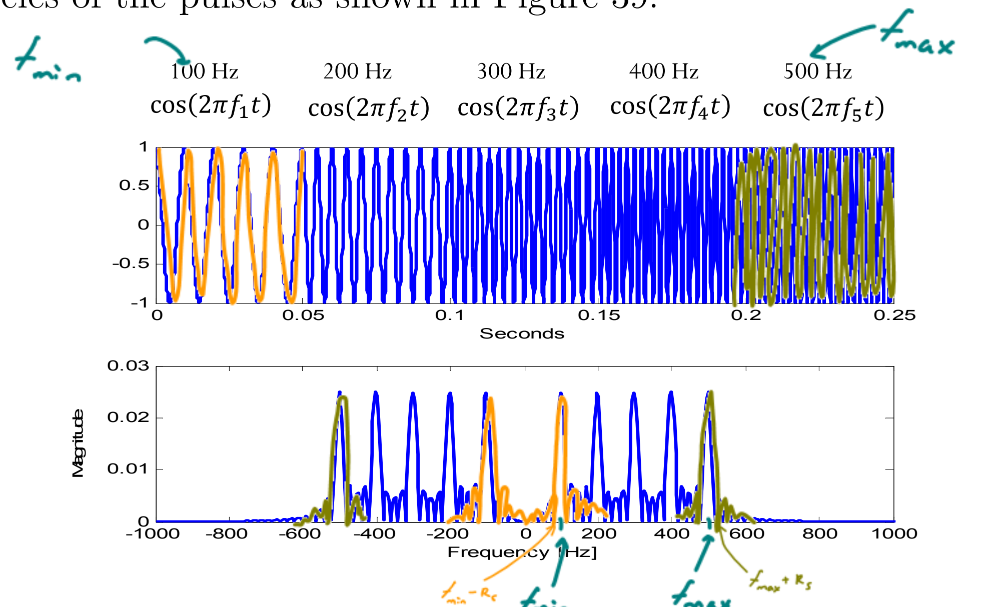


Figure 39: A digital version of FM: $x_{\text{FM}}(t)$ and the corresponding $X_{\text{FM}}(f)$.

- $X(f)$ extends to $\pm\infty$. It is not band-limited.
- One may approximate its bandwidth by assuming that "most" of the energy in the sinc function is contained in its main lobe which is at $\pm\frac{1}{T_s} = \pm R_s$ from its peak. Therefore, the bandwidth of $x_{\text{FM}}(t)$ becomes

$$\mathrm{BW}_{\text{FM}} \approx R_s + (f_{\max} - f_{\min}) + R_s = \underbrace{(f_{\max} - f_{\min})}_{2k_f m_p} + 2R_s$$

For FM,

$f(t) = f_c + k_f \underbrace{m(t)}_{\in(-m_p, +m_p)}$

$f_{max} = f_c + k_f m_p$

$f_{min} = f_c - k_f m_p$

Recall, for NBFM, $k_f \approx 0$ and BW $= 2B$

So, the general formula for BW_{FM} is

$2k_f m_p + 2B$ ← Carson's formula